Exercise 1: Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} 0, & (x,y) = (0,0) \\ \dfrac{xy(x^2-y^2)}{x^2+y^2}, & (x,y) \neq (0,0). \end{cases}$$

(a) (0.5 points) Do all directional derivatives of f at $(x,y) = (0,0)$ exist? Evaluate the directional derivatives whenever they exist.

(b) (0.5 points) Calculate $D_1 f$ and $D_2 f$ at $(x,y) \neq (0,0)$.

(c) (0.5 points) Show that $D_1 f$ and $D_2 f$ exist at $(x,y) = (0,0)$.

(d) (1 point) Is f differentiable at $(x,y) = (0,0)$? Justify your answer.

(e) (0.5 points) Is f continuous at $(x,y) = (0,0)$? Justify your answer.

(f) (0.5 points) Show that $D_2 D_1 f$ and $D_1 D_2 f$ exist at $(x,y) = (0,0)$, but that they are not equal there. What does this tell us about the differentiability class of f?

a) let $u = (h,k) \neq (0,0)$. We evaluate:

$$\frac{f(0+tu)-f(0)}{t} = \frac{f(th,tk)}{t} = \frac{t^2hk(t^2h-t^2k)}{t(t^2h^2+t^2k^2)} = \frac{t^4hk(h-k)}{t^3(h^2+k^2)} = t\,\frac{hk(h-k)}{h^2+k^2}$$

The limit as $t \to 0$ of the above function is zero.

That is, all directional derivatives at $(x,y) = (0,0)$ are zero.

b) $$D_1 f = \frac{[y(x^2-y^2)+xy(2x)](x^2+y^2) - 2x(xy(x^2-y^2))}{(x^2+y^2)^2} = \frac{(3x^2y-y^3)(x^2+y^2)-2x^2y(x^2-y^2)}{(x^2+y^2)^2}$$

$$= \frac{3x^4y+3x^2y^3-x^2y^3-y^5-2x^4y+2x^2y^3}{(x^2+y^2)^2} = \frac{x^4y+4x^2y^3-y^5}{(x^2+y^2)^2} = y\,\frac{x^4+4x^2y^2-y^4}{(x^2+y^2)^2}$$

$$D_2 f = \frac{[x(x^2-y^2)-2y(xy)](x^2+y^2)-2y(xy(x^2-y^2))}{(x^2+y^2)^2} = \frac{[x^3-3xy^2](x^2+y^2)-2x^3y^2+2xy^4}{(x^2+y^2)^2}$$

$$= \frac{x^5+x^3y^2-3x^3y^2-3xy^4-2x^3y^2+2xy^4}{(x^2+y^2)^2} = \frac{x^5-4x^3y^2-xy^4}{(x^2+y^2)^2} = x\,\frac{x^4-4x^2y^2-y^4}{(x^2+y^2)^2}$$

c) Taking $u = \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}$ in a), we see that $D_1 f(0) = D_2 f(0) = 0$.

d) Yes because the partial derivatives are continuous at $(x,y) = (0,0)$

Indeed, the numerator is of degree 5. Thererore: $\lim\limits_{(x,y)\to(0,0)} y\,\dfrac{x^4+4x^2y^2-y^4}{(x^2+y^2)^2} = 0$, similarly for $D_2 f$.

Alternatively: $\lim\limits_{t\to 0} tb\,\dfrac{t^4a^4+4t^4a^2b^2-t^4b^4}{t^4(a^2+b^2)^2} = 0$

e) Yes because f is differentiable at $(x,y) = (0,0)$.

f) $$\left.\begin{aligned} D_2 D_1 f &= \lim_{t\to 0} \frac{D_1 f(0+te_2) - D_1 f(0)}{t} = \frac{-t^5}{t^5} = -1 \\ D_1 D_2 f &= \lim_{t\to 0} \frac{D_2 f(0+te_1) - D_2 f(0)}{t} = \frac{t^5}{t^5} = 1 \end{aligned}\right\}$$ This means that f is <u>not</u> C^2

Exercise 2: (1.5 points) Let $g : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by the equation $g(x,y) = \big(x, y + x^2\big)$. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined as

$$f(x,y) = \begin{cases} 0, & (x,y) = (0,0), \\ \dfrac{x^2 y}{x^4 + y^2}, & (x,y) \neq (0,0). \end{cases}$$

Let $h = f \circ g$. Show that the directional derivatives of f and g exist everywhere, but that there is a vector $\vec{\mathbf{u}} \neq \mathbf{0}$ for which the directional derivative of h in the direction of $\vec{\mathbf{u}}$, that is $\mathrm{D}h(\mathbf{0})\vec{\mathbf{u}}$, does not exist.

For g, we simply notice that its components are polynomials.

For f: let $p = (a,b) \in \mathbb{R}^2$, $u = (h,k) \neq 0 \in \mathbb{R}^2$ and $t \in \mathbb{R}$. We evaluate:

$$\frac{f(p+tu) - f(p)}{t} = \frac{f(a+th, b+tk) - f(a,b)}{t} = \frac{1}{t}\left(\frac{(a+th)^2(b+tk)}{(a+th)^4 + (b+tk)^2} - \frac{a^2 b}{a^4 + b^2} \right)$$

$$\frac{1}{t}\left(\underbrace{\frac{(a+th)^2(b+tk)(a^4+b^2) - a^2 b\left[(a+th)^4 + (b+tk)^2\right]}{\left[(a+th)^4 + (b+tk)^2\right](a^4+b^2)}}_{O(t)} \right), \quad \text{So: } \lim_{t\to 0} \frac{f(p+tu) - f(p)}{t} = 0$$

$$h = f\circ g = f(x, y+x^2) = \begin{cases} 0, & (x,y) = (0,0) \\ \dfrac{x^2(y+x^2)}{x^4 + (y+x^2)^2}, & (x,y) \neq (0,0) \end{cases}$$

Let $u = (a,b)$, then we evaluate

$$\frac{h(ta, tb) - h(0,0)}{t} = \frac{1}{t} \frac{t^2 a^2 (tb + t^2 a^2)}{t^4 a^4 + (tb + t^2 a^2)^2} = \frac{t^2 a^2 (b + ta)}{t^4 a^2 + t^2 b^2 + 2t^3 b a^2 + t^4 a^4}$$

$$= \frac{a^2 (b + ta)}{b^2 + 2tba^2 + 2t^2 a^2}$$

its limit as $t \to 0$ is $\frac{a^2}{b}$, therefore, the directional derivative does not exist along $\vec{u} = \begin{pmatrix} a \\ 0 \end{pmatrix}$

Exercise 3: Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by the equation

$$f(x,y) = \left(x^2 - y^2, 2xy\right).$$

Let $A = \left\{(x,y) \in \mathbb{R}^2 \,|\, x > 0\right\}$.

(a) (0.5 points) Show that f is one-to-one on the set A. [Hint: you may use the fact that if $f(x,y) = f(a,b)$, then $\|f(x,y)\| = \|f(a,b)\|$ (where $\|\cdot\|$ denotes the Euclidean norm) to find a contradiction.]

(b) (1 point) If g is the inverse function, find $\mathrm{D}g(0,1)$.

a) Suppose $(a,b) \neq (x,y)$ and $f(x,y) = f(a,b)$

Then, it must hold that $\|f(x,y)\| = \|f(a,b)\|$

$$\to \sqrt{(x^2-y^2)^2 + 4x^4y^4} = \sqrt{(a^2-b^2)^2 + 4a^4b^4} \leftrightarrow \sqrt{(x^2+y^2)^2} = \sqrt{(a^2+b^2)^2} \leftrightarrow x^2+y^2 = a^2+b^2$$

Let then $b = \pm\sqrt{x^2+y^2-a^2}$

$$f(x,y) = f(a,b) \leftrightarrow \left(x^2-y^2, 2xy\right) = \left(a^2 - (x^2+y^2-a^2), \pm 2a\sqrt{x^2+y^2-a^2}\right)$$

$\leftrightarrow \left(x^2-y^2, 2xy\right) = \left(-x^2-y^2, \pm 2a\sqrt{x^2+y^2-a^2}\right)$ but this requires $a=0$, which is a contradiction.

b) $Df = \begin{bmatrix} 2x & -2y \\ 2y & 2x \end{bmatrix}$, $\det(Df) = 4x^2 + 4y^2 \neq 0 \quad \forall \ (x,y) \in A$

we also know that f is one-to-one in A. So $\left(x^2-y^2, 2xy\right) = (0,1) \to \begin{cases} x^2 = y^2 \\ 2xy = 1 \to y = \frac{1}{2x} \to x^2 = \frac{1}{4x^2} \to x = \frac{1}{\sqrt{2}} \end{cases}$

That is $(x,y) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$$\hookrightarrow f\underbrace{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)}_{\text{"a"}} = \underbrace{(0,1)}_{\text{"b"}}$$

Now we can compute: $Dg(y) = \left[Df(g(y))\right]^{-1}$, $g(y) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$$Df(g(y)) = \begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{bmatrix} \to Dg(0,1) = \frac{1}{4}\begin{bmatrix} \sqrt{2} & \sqrt{2} \\ -\sqrt{2} & \sqrt{2} \end{bmatrix} = \frac{1}{2\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

Exercise 4: (1 point) Let $f : \mathbb{R}^{k+n} \to \mathbb{R}^n$ be of class C^1; suppose that $f(\mathbf{a}) = \mathbf{0}$ and that $Df(\mathbf{a})$ has rank n. Show that if $\mathbf{c}$ is a point of $\mathbb{R}^n$ sufficiently close to $\mathbf{0}$, then the equation $f(\mathbf{x}) = \mathbf{c}$ has a solution.

Let $x = (u,v)$, $u \in \mathbb{R}^k$, $v \in \mathbb{R}^n$, we write $a = (a_u, a_v)$ and $Df(a) = [A \quad B]$, $A \in \mathbb{R}^{n\times k}$, $B \in \mathbb{R}^{n\times n}$

with B full-rank

Next, let $F(u,v) = (u, f(u,v)) \to DF(a) = \begin{bmatrix} I & 0 \\ A & B \end{bmatrix}$, which is invertible.

$$F(a) = F(a_u, a_v) = (a_u, f(a)) = (a_u, 0)$$

This is the important observation!

So, by the inverse function theorem $\exists!\, G : \mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R}^k \times \mathbb{R}^n : F(G(z)) = z$, for $z \in B_R(a_u, 0)$

From the form of F, we further know that $G(z) = (z_1, g(z_1, z_2))$

Therefore: $(a_u, c) = F(G(a_u,c)) = F(a_u, g(a_u,c)) = (a_u, f(a_u, g(a_u,c)))$

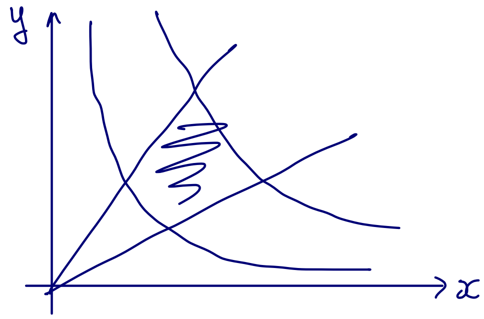
and so $x = (a_u, g(a_u,c))$ is the solution.

Exercise 5: (0.5 points) Let $\mathcal{S}$ be a closed curve in $\mathbb{R}^2$ and $\mathcal{C}$ the unit circle in $\mathbb{R}^2$. Suppose that $\mathcal{S}$ and $\mathcal{C}$ are diffeomorphic. What is the 2-dimensional volume of the curve $\mathcal{S}$ ($\operatorname{vol}_2 \mathcal{S}$)? Justify your answer in full detail.

Hints and remarks: for this exercise you may assume that "$\mathcal{S}$ and $\mathcal{C}$ are diffeomorphic" means that there is a C^r-function f, $r \geq 1$, with C^r inverse, such that $f : \mathcal{S} \to \mathcal{C}$ and $f^{-1} : \mathcal{C} \to \mathcal{S}$.

Here we only recall that the volume is scaled via the diffeomorphism. But since $\operatorname{vol}_2 C = 0 \rightarrow \operatorname{vol}_2 S = 0$.

Exercise 6: (1 point) Let B be the portion of the first quadrant in $\mathbb{R}^2$ lying between the hyperbolas $xy = 1$ and $xy = 2$ and the two straight lines $y = x$ and $y = 4x$. Evaluate $\int_B x^2y^3 \mathrm{d}x\mathrm{d}y$. [Hint: use the change of variables $x = u/v$ and $y = uv$.]



$$x = \frac{u}{v} , \quad y = uv$$

$$\Phi(u,v) = \left(\frac{u}{v}, uv\right) = (x,y)$$

$$D\Phi = \begin{bmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ v & u \end{bmatrix}$$

$$\det(D\Phi) = \frac{u}{v} + \frac{u}{v} = 2\frac{u}{v}$$

$$xy = \left(\frac{u}{v}\right)uv = u^2 \in [1, 2] \rightarrow u \in [1, \sqrt{2}]$$

$$\left.\begin{aligned} y = x &\rightarrow \frac{u}{v} = uv \longrightarrow v^2 = 1 \\ y = 4x &\rightarrow \frac{u}{v} = 4uv \rightarrow v^2 = \frac{1}{4} \end{aligned}\right\} \rightarrow v \in \left[\frac{1}{2}, 1\right]$$

Lecture 9 (change of variables)

$$\int_B x^2y^3\,dx\,dy = \int_{1/2}^{1}\int_{1}^{\sqrt{2}} \left(\frac{u}{v}\right)^2 (uv)^3 \left(2\frac{u}{v}\right) du\,dv = 2\int_{\frac{1}{2}}^{1}\int_{1}^{\sqrt{2}} u^6 v\,du\,dv$$

$$= 2\int_{1/2}^{1} \left.\frac{u^7}{7}\right|_{1}^{\sqrt{2}} v\,dv = 2\int_{\frac{1}{2}}^{1} \left(\frac{2^{7/2}}{7} - \frac{1}{7}\right) v\,dv = \frac{2^{7/2} - 1}{7}$$

Exercise 7: Let $U = \mathbb{R}^2 \backslash \mathbf{0}$; consider the 1-form in A defined by the equation

$$\omega = \frac{x\,\mathrm{d}x + y\,\mathrm{d}y}{x^2 + y^2}.$$

(a) (1 point) Show that ω is closed.

(b) (1 point) Show that ω is exact on U.

a) $w = \frac{x}{x^2+y^2}\,dx + \frac{y}{x^2+y^2}\,dy$

$$dw = \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right) dy \wedge dx + \frac{\partial}{\partial x}\left(\frac{y}{x^2+y^2}\right) dx \wedge dy = \left(\frac{-2xy}{(x^2+y^2)^2}\right) dy \wedge dx + \left(\frac{-2xy}{(x^2+y^2)^2}\right) dx \wedge dy = 0$$

b) w is exact if there is a function $f = f(x,y)$: $w = df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$

$x = r\cos\theta$, $y = r\sin\theta$

$dx = dr\cos\theta - r\sin\theta\,d\theta$, $dy = dr\sin\theta + r\cos\theta\,d\theta$

$$\tilde{\omega} = \frac{r\cos\theta\,(dr\cos\theta - r\sin\theta\,d\theta) + r\sin\theta\,(dr\sin\theta + r\cos\theta\,d\theta)}{r^2} = \frac{r\,dr\cos^2\theta - r^2\cancel{\sin\theta\cos\theta}\,d\theta + r\,dr\sin^2\theta + r^2\cancel{\sin\theta\cos\theta}\,d\theta}{r^2}$$

$$\tilde{\omega} = \frac{dr}{r} \quad ; \quad \tilde{f} = \tilde{f}(r,\theta) : d\tilde{f} = \frac{\partial\tilde{f}}{\partial r}dr + \frac{\partial\tilde{f}}{\partial\theta}d\theta = \frac{dr}{r} \rightarrow \tilde{f} = \ln(r)$$

$$\rightarrow \boxed{f = \ln\left(\sqrt{x^2+y^2}\right)}$$

Exercise 8: (1 point) Let C be a closed curve in the plane. Show that $\begin{bmatrix} xy^2 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -x^2y \end{bmatrix}$ do equal work around C.

$F_1 = \begin{bmatrix} xy^2 \\ 0 \end{bmatrix}$, $F_2 = \begin{bmatrix} 0 \\ -x^2y \end{bmatrix}$

$W_{F_1} = xy^2\,dx$, $W_{F_2} = -x^2y\,dy$

$dW_{F_1} = 2xy\,dy\wedge dx$, $dW_{F_2} = -2xy\,dx\wedge dy = -dW_{F_1}$. It follows from Stokes theorem

that : $\int_C W_{F_1} = \int_S dW_{F_1} = \int_S dW_{F_2} = \int_C W_{F_2}$

Exercise 9: (1 point) What is the integral $\int_S x_3 \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_4$, where S is the part of the three-dimensional manifold of equation

$$x_4 = x_1 x_2 x_3 \quad \text{where } 0 \le x_1, x_2, x_3 \le 1,$$

oriented by $\Omega = \operatorname{sgn} \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3$? [Hint: This surface is a graph, so it is easy to parametrize.]

Tutorial 12 exercise 8